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A HIGHER-ORDER LINEAR THEORY FOR ISOTROPIC PLATES—I. THEORETICAL CONSIDERATIONS

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Abstract—Using estimates of the strain field a new refined version of the linear theory of isotropic plates is constructed. The theory is expressed in terms of the same independent parameters of displacements as in the first-order shear deformation theory. The accuracy of the strain energy in this theory is of the order $(h/L)^4$ (h—thickness, L—wavelength of deformation patterns) whereas the classical theory of plates has accuracy of the order $(h/L)^2$. The results of the present analysis provide evidence that some of the earlier proposed higher-order theories of isotropic plates are based on an incorrect hypothesis for the displacement field.

1. INTRODUCTION

Recently, many different higher-order theories of elastic plates have been proposed. The formulation of these theories, which are aimed at not-so-thin plates or at stress concentration problems, has been based upon *ad hoc* assumptions for the displacement field or for the stress field. Some examples of such theories are presented in the papers by Mushtari (1959), Kączkowski (1968), Jemielita (1975), Reissner (1975), Lo *et al.* (1977, 1978), Levinson (1980), Reissner (1983), Preusser (1984), Lewiński (1987) and Chen and Archer (1989). For details and for more complete lists of references see the reviews by Reissner (1985) and by Lewiński (1986, 1987). In a similar way, refined theories for the case of composite laminated plates have been developed. An account of the generalization of theories for this type of plate was given by Librescu and Reddy (1989).

In the theories by Mushtari (1959), Kączkowski (1968), Jemielita (1975), Reissner (1975), Levinson (1980) and Lewiński (1987) the displacement field for the in-plane displacements is assumed to be of the form (z—out-plane coordinate)

$$u_\alpha = \overset{0}{u}_\alpha + z\overset{1}{u}_\alpha + z^3/6\overset{3}{u}_\alpha, \quad (\alpha = 1, 2).$$

Next, using the conditions that the transverse shear stresses should vanish on the plate top and bottom surfaces and assuming that the normal strain e_{zz} is equal to zero (or an equivalent hypothesis), the parameters $\overset{3}{u}_\alpha$ are expressed by the shearing strains in the normal direction. Although it leads to simplified theories which seem to be useful for practical purposes some corrections in the basic hypotheses are needed.

In this paper we abandon assumptions for the displacement or stress field. A linear theory of isotropic plates is generated by means of estimations in a global sense. Starting from three-dimensional equations of elasticity (Section 2) the estimations for components of the two-dimensional strain field are established (Section 3). In Section 4 we show a method for the construction of a simple refined theory of plates which has only five independent parameters of the displacement field and which differs from those proposed earlier. Furthermore, we give clearly the conditions which have to be met to ensure the accuracy of the plate theory. Finally, we point out that in order to construct the consistent higher-order theory the hypothesis $e_{zz} = 0$ (or an equivalent hypothesis) should not be used. The correct relation which allows the elimination of the parameters $\overset{3}{u}_\alpha$ is presented.

For simplicity's sake in the analysis of estimations we omit consideration of loads distributed over the faces of the plate. Moreover, our analysis is restricted to the regular boundary conditions (Koiter, 1970).

2. BASIC EQUATIONS

Let x^α be an orthogonal curvilinear coordinate system on the middle surface of a plate and $x^3 = z$ denotes normal distance from this surface.

With reference to this coordinate system, the field equations of the linear theory of elasticity constitute the following set of equations (Greek indices range over 1, 2; Latin indices range over 1, 2, 3, unless specifically noted otherwise):

the strain–displacement relations

$$e_{\alpha\alpha} = u_{\alpha,\alpha}/A_\alpha - u_\beta/\rho_\alpha, \quad (\alpha \neq \beta), \tag{1}$$

$$e_{12} = B_1^+ u_2 + B_2^+ u_1, \tag{2}$$

$$e_{\alpha 3} = u_{3,\alpha}/A_\alpha + u_{\alpha,3}, \tag{3}$$

$$e_{33} = u_{3,3}, \tag{4}$$

where

$$B_\alpha^+ f = f_{,\alpha}/A_\alpha + f/\rho_\beta, \quad (\alpha \neq \beta), \tag{5}$$

$$1/\rho_\alpha = -A_{\alpha,\beta}/(A_\alpha A_\beta), \quad (\alpha \neq \beta), \tag{6}$$

e_{ij} are physical components of the strain tensor, u_i are components of the displacement vector U, A_α are the Lamé coefficients and ρ_α are radii of curvature of the lines x^2 = constant and x^1 = constant, respectively;

the stress–strain relations

$$\sigma_{\alpha\alpha} = \frac{2G}{1-\nu}(e_{\alpha\alpha} + \nu e_{\beta\beta}) + \frac{\nu}{1-\nu}\sigma_{33}, \quad (\alpha \neq \beta), \tag{7}$$

$$\sigma_{\alpha j} = G e_{\alpha j}, \quad (\alpha \neq j), \tag{8}$$

$$e_{33} = -\frac{\nu}{1-\nu}(e_{11} + e_{22}) + (1-2\nu)/[2G(1-\nu)]\sigma_{33}, \tag{9}$$

where σ_{ij} are physical components of the stress tensor, $G = E/[2(1+\nu)]$, E is the Young's modulus, ν is Poisson's ratio;

the equilibrium equations

$$\sigma_{\alpha\alpha,\alpha}/A_\alpha - (\sigma_{\alpha\alpha} - \sigma_{\beta\beta})/\rho_\beta + \sigma_{\alpha\beta,\beta}/A_\beta - 2\sigma_{\alpha\beta}/\rho_\alpha + \sigma_{\alpha 3,3} + X_\alpha = 0, \quad (\alpha \neq \beta), \tag{10}$$

$$B_1^- \sigma_{13} + B_2^- \sigma_{23} + \sigma_{33,3} + X_3 = 0, \tag{11}$$

where

$$B_\alpha^- f = f_{,\alpha}/A_\alpha - f/\rho_\beta, \quad (\alpha \neq \beta), \tag{12}$$

X_α and X_3 are components of body forces per unit volume.

Apart from these equations we shall need an additional relation which will be used to transform eqn (10) into an alternative form. The derivation of this relation is shown below.

After differentiation of the displacement vector $U = u_1 g_1 + u_2 g_2 + u_3 n$ and making use of the strain–displacement relations (1)–(4), we may derive the following relations:

$$U_{,\alpha} = A_\alpha[e_{\alpha\alpha} g_\alpha + (e_{12}/2 + \Omega_\alpha) g_\beta + (e_{\alpha 3} - u_{\alpha,3}) n], \quad (\alpha \neq \beta), \tag{13}$$

where

$$\Omega_\alpha = \begin{cases} \Omega, & \alpha = 1, \\ -\Omega, & \alpha = 2, \end{cases} \tag{14}$$

$$\Omega = \tfrac{1}{2}(B_1^- u_2 - B_2^- u_1). \tag{15}$$

The unit vectors (g_1, g_2, n) represent the base vectors of the coordinate system (x^α, z), and Ω is the rotation about the normal n.

Since the order of common partial differentiations is interchangeable we can write

$$U_{,\alpha\beta} = U_{,\beta\alpha}, \quad (\alpha \neq \beta). \tag{16}$$

When we differentiate (13) with respect to x^β and make use of (16), we find that

$$e_{\alpha\beta,\beta}/A_\beta - 2e_{\alpha\beta}/\rho_\alpha = 2[e_{\beta\beta,\alpha}/A_\alpha - (e_{\alpha\alpha} - e_{\beta\beta})/\rho_\beta - \Omega_{\alpha,\beta}/A_\beta], \quad (\alpha \neq \beta). \tag{17}$$

With the help of (17) and the stress–strain relations (7)–(9) the alternative forms of the equilibrium equation (10) become

$$e_{\alpha 3,3} = -\frac{2}{1-\nu}(e_{11} + e_{22})_{,\alpha}/A_\alpha - \frac{\nu}{(1-\nu)G}\sigma_{33,\alpha}/A_\alpha + 2\Omega_{\alpha,\beta}/A_\beta - X_\alpha/G, \quad (\alpha \neq \beta), \tag{18}$$

and

$$0.5\nu e_{\alpha 3,3} = e_{33,\alpha}/A_\alpha - \frac{1-\nu}{2G}\sigma_{33,\alpha}/A_\alpha + \nu\Omega_{\alpha,\beta}/A_\beta - 0.5\nu X_\alpha/G, \quad (\alpha \neq \beta). \tag{19}$$

Let the displacements, the strains, the rotation about the normal vector and the stresses be represented as series with respect to z:

$$u_i = \overset{0}{u}_i + z/1!\overset{1}{u}_i + \cdots + z^k/k!\overset{k}{u}_i + \cdots, \tag{20}$$

$$e_{ij} = \overset{0}{\varepsilon}_{ij} + z/1!\overset{1}{\varepsilon}_{ij} + \cdots + z^k/k!\overset{k}{\varepsilon}_{ij} + \cdots, \tag{21}$$

$$\Omega = \overset{0}{\omega} + z/1!\overset{1}{\omega} + \cdots + z^k/k!\overset{k}{\omega} + \cdots, \tag{22}$$

$$\sigma_{ij} = \overset{0}{\sigma}_{ij} + z/1!\overset{1}{\sigma}_{ij} + \cdots + z^k/k!\overset{k}{\sigma}_{ij} + \cdots. \tag{23}$$

When we apply (20)–(22) to the strain–displacement relations (1)–(4) and to eqn (15), we obtain the two-dimensional relations

$$\overset{k}{\varepsilon}_{\alpha\alpha} = \overset{k}{u}_{\alpha,\alpha}/A_\alpha - \overset{k}{u}_\beta/\rho_\alpha, \quad (\alpha \neq \beta) \tag{24}$$

$$\overset{k}{\varepsilon}_{12} = B_1^+ \overset{k}{u}_2 + B_2^+ \overset{k}{u}_1, \tag{25}$$

$$\overset{k}{\varepsilon}_{\alpha 3} = \overset{k}{u}_{3,\alpha}/A_\alpha + \overset{k+1}{u}_\alpha, \tag{26}$$

$$\overset{k}{\varepsilon}_{33} = \overset{k+1}{u}_3, \tag{27}$$

$$\overset{k}{\omega} = \tfrac{1}{2}(B_1^- \overset{k}{u}_2 - B_2^- \overset{k}{u}_1). \tag{28}$$

In the above and the following formulae the index k takes the values $0, 1, \ldots,$ unless otherwise stated.

The equation (26) may be written in the alternative form

$$\overset{k+1}{u}_\alpha = \overset{k}{\varepsilon}_{\alpha 3} - \overset{k}{u}_{3,\alpha}/A_\alpha, \tag{29}$$

or after using (27)

$$\overset{k+2}{u}_\alpha = \overset{k+1}{\varepsilon}_{\alpha 3} - \overset{k}{\varepsilon}_{33,\alpha}/A_\alpha. \tag{30}$$

By inserting the right-hand side of (29) into (28), we find that

$$\overset{k+1}{\omega} = \tfrac{1}{2}(B_1^- \overset{k}{\varepsilon}_{23} - B_2^- \overset{k}{\varepsilon}_{13}). \tag{31}$$

When we substitute the right-hand side of (30) into (24) and (25), we have

$$\overset{k+2}{\varepsilon}_{\alpha\alpha} = (\overset{k+1}{\varepsilon}_{\alpha 3} - \overset{k}{\varepsilon}_{33,\alpha}/A_\alpha)_{,\alpha}/A_\alpha - (\overset{k+1}{\varepsilon}_{\beta 3} - \overset{k}{\varepsilon}_{33,\beta}/A_\beta)/\rho_\alpha, \quad (\alpha \neq \beta), \tag{32}$$

$$\overset{k+2}{\varepsilon}_{12} = B_1^+ (\overset{k+1}{\varepsilon}_{23} - \overset{k}{\varepsilon}_{33,2}/A_2) + B_2^+ (\overset{k+1}{\varepsilon}_{13} - \overset{k}{\varepsilon}_{33,1}/A_1). \tag{33}$$

The two-dimensional form of eqn (9) becomes

$$\overset{k}{\varepsilon}_{33} = -\frac{\nu}{1-\nu}(\overset{k}{\varepsilon}_{11} + \overset{k}{\varepsilon}_{22}) + (1-2\nu)/[2G(1-\nu)]\overset{k}{\sigma}_{33}. \tag{34}$$

Integrating eqn (18) we obtain

$$e_{\alpha 3} = \int f_\alpha \, dz + C, \tag{35}$$

where f_α represents the right-hand side of (18) and C is an arbitrary constant.

The function f_α can be written in the form of a series

$$f_\alpha = \overset{0}{f}_\alpha + z/1!\overset{1}{f}_\alpha + \cdots, \tag{36}$$

in which $\overset{0}{f}_\alpha, \overset{1}{f}_\alpha, \ldots$ do not depend on z.

Using the condition that the transverse shear strains $e_{\alpha 3}$ should vanish on the plate top ($z = h/2$) and bottom ($z = -h/2$) surfaces, where h is the thickness, yields

$$e_{\alpha 3} = -[h^2/(2!2^2)\overset{1}{f}_\alpha + h^4/(4!2^4)\overset{3}{f}_\alpha + \cdots] + z/1!\overset{0}{f}_\alpha + z^2/2!\overset{1}{f}_\alpha + \cdots, \tag{37}$$

where

$$\overset{k+1}{\varepsilon}_{\alpha 3} = \overset{k}{f}_{\alpha} = -\frac{2}{1-\nu}(\overset{k}{\varepsilon}_{11}+\overset{k}{\varepsilon}_{22})_{,\alpha}/A_{\alpha} - \frac{\nu}{(1-\nu)G}\overset{k}{\sigma}_{33,\alpha}/A_{\alpha} + 2\overset{k}{\omega}_{\alpha,\beta}/A_{\beta} - \overset{k}{X}_{\alpha}/G, \quad (\alpha \neq \beta), \tag{38}$$

$$\overset{k}{\omega}_{\alpha} = \begin{cases} \overset{k}{\omega}, & \alpha = 1, \\ -\overset{k}{\omega}, & \alpha = 2. \end{cases} \tag{39}$$

$\overset{k}{X}_{\alpha}$ is defined by

$$X_{\alpha} = \overset{0}{X}_{\alpha} + z/1!\overset{1}{X}_{\alpha} + \cdots. \tag{40}$$

Similarly, we integrate eqn (19). The result is

$$0.5\nu e_{\alpha 3} = -[h^2/(2!2^2)\overset{1}{f}{}^*_{\alpha} + h^4/(4!2^4)\overset{3}{f}{}^*_{\alpha} + \cdots] + z/1!\overset{0}{f}{}^*_{\alpha} + z^2/2!\overset{1}{f}{}^*_{\alpha} + \cdots, \tag{41}$$

where

$$0.5\nu\overset{k+1}{\varepsilon}_{\alpha 3} = \overset{k}{f}{}^*_{\alpha} = \overset{k}{\varepsilon}_{33,\alpha}/A_{\alpha} - \frac{1-\nu}{2G}\overset{k}{\sigma}_{33,\alpha}/A_{\alpha} + \nu\overset{k}{\omega}_{\alpha,\beta}/A_{\beta} - \nu/2\overset{k}{X}_{\alpha}/G, \quad (\alpha \neq \beta). \tag{42}$$

The equilibrium equation (11) can be written, after dividing by G

$$\sigma_{33,3}/G = -B_1^- e_{13} - B_2^- e_{23} - X_3/G. \tag{43}$$

Following the method used above, we also have

$$\sigma_{33}/G = -[h^2/(2!2^2)\overset{1}{f}_3 + h^4/(4!2^4)\overset{3}{f}_3 + \cdots] + z/1!\overset{0}{f}_3 + z^2/2!\overset{1}{f}_3 + \cdots, \tag{44}$$

where

$$\overset{k+1}{\sigma}_{33}/G = \overset{k}{f}_3 = -B_1^-\overset{k}{\varepsilon}_{13} - B_2^-\overset{k}{\varepsilon}_{23} - \overset{k}{X}_3/G. \tag{45}$$

$\overset{k}{X}_3$ is defined by

$$X_3 = \overset{0}{X}_3 + z/1!\overset{1}{X}_3 + \cdots. \tag{46}$$

All the relations just obtained will be used to estimate the components of the strain field.

3. ESTIMATES

It is well known that in flexible bodies like plates or shells the transverse stresses are of a lower order than the stresses parallel to the middle surface (Koiter, 1959). The concrete estimates of the magnitudes of the stresses $\sigma_{\alpha 3}$ and σ_{33} are the following (John, 1965; Koiter, 1967):

$$\sigma_{\alpha 3} = 0(E\eta\varepsilon), \tag{47}$$

$$\sigma_{33} = 0(E\eta^2\varepsilon). \tag{48}$$

The quantity η represents the small parameter defined by

$$\eta = h/L, \tag{49}$$

in which L is the smallest wavelength of deformation patterns. The definition of L is given by the relations

$$\max_{x^1, x^2} \left(|f_{,\alpha}/A_\alpha|, |B_\alpha^+ f|, |B_\alpha^- f| \right) \leqslant |f|/L, \tag{50}$$

where f is a differentiable function.

The quantity ε is the maximum strain parallel to the middle surface and it is specified by

$$\varepsilon = \max_{x^1, x^2} \left(|\overset{0}{\varepsilon}_{\alpha\beta}|, h|\overset{1}{\varepsilon}_{\alpha\beta}| \right), \quad (\alpha, \beta = 1, 2). \tag{51}$$

In order to determine the estimates of the two-dimensional components of strains we make use of the relations (47) and (48). In addition, for the body forces we assume that

$$\overset{0}{X}_\alpha/G = 0(\eta\varepsilon/h), \tag{52}$$

$$\overset{k}{X}_\alpha/G = 0(\eta^3\varepsilon/h^{k+1}), \quad k = 1, 2, \ldots, \tag{53}$$

$$\max_{x^1, x^2} \left(|\overset{0}{X}_{\alpha,\beta}/A_\beta|, \quad |\overset{0}{X}_\alpha/\rho_\beta| \right) = 0(G\eta^4\varepsilon/h^2), \quad (\alpha, \beta = 1, 2), \tag{54}$$

$$\overset{k}{X}_3/G = 0(\eta^2\varepsilon/h^{k+1}), \quad k = 0, 1, \ldots, \tag{55}$$

and for the rotation about the normal vector n we assume that

$$\overset{0}{\omega}_{,\alpha}/A_\alpha = 0(\eta\varepsilon/h), \tag{56}$$

$$\max_{x^1, x^2} \left(|(\overset{0}{\omega}_{,\beta}/A_\beta)_{,\alpha}/A_\alpha|, \quad |1/\rho_\alpha \overset{0}{\omega}_{,\beta}/A_\beta| \right) = 0(\eta^4\varepsilon/h^2), \quad (\alpha, \beta = 1, 2). \tag{57}$$

After dividing both sides of (47) and (48) by G and setting $z = 0$, we obtain

$$\overset{0}{\varepsilon}_{\alpha 3} = 0(E/G\eta\varepsilon), \tag{58}$$

$$\overset{0}{\sigma}_{33} = 0(E/G\eta^2\varepsilon). \tag{59}$$

Making use of the well known inequality $|a+b| \leqslant |a|+|b|$, we bring eqn (45) into the following relation:

$$|\overset{1}{\sigma}_{33}/G| \leqslant |B_1^- \overset{0}{\varepsilon}_{13}| + |B_2^- \overset{0}{\varepsilon}_{23}| + |\overset{0}{X}_3/G|. \tag{60}$$

By means of (50) we can write the inequality

$$h|B_\alpha^- \overset{0}{\varepsilon}_{\alpha 3}| \leqslant h/L|\overset{0}{\varepsilon}_{\alpha 3}|, \tag{61}$$

and with the aid of (49) and (58) we arrive at the following estimate

$$hB_\alpha^- \overset{0}{\varepsilon}_{\alpha 3} = 0(\eta^2\varepsilon). \tag{62}$$

Finally, making use of (55) yields

$$h\overset{1}{\sigma}_{33}/G = 0(\eta^2\varepsilon). \tag{63}$$

Similarly, we can derive from (31) the following relation:

$$h\overset{1}{\omega} = 0(\eta^2\varepsilon). \tag{64}$$

By the use of (38) we can estimate $\overset{1}{\varepsilon}_{\alpha 3}$ and $\overset{2}{\varepsilon}_{\alpha 3}$. It follows from (50), (51), (59), (63), (56), (64), (52) and (53) that the result of the estimation is

$$h^k\overset{k}{\varepsilon}_{\alpha 3} = 0(\eta\varepsilon), \quad k = 1, 2. \tag{65}$$

Assuming first $k = 0$ and then $k = 1$ in (34) and taking into account (51), (59) and (63), we find

$$h^k\overset{k}{\varepsilon}_{33} = 0(\varepsilon), \quad k = 0, 1. \tag{66}$$

Similarly, assuming first $k = 0$ and then $k = 1$ in (32) and (33) and taking into account (65) and (66), we have

$$h^k\overset{k}{\varepsilon}_{\alpha\beta} = 0(\eta^2\varepsilon), \quad k = 2, 3, \quad (\alpha, \beta = 1, 2). \tag{67}$$

Considering again eqns (45) and (31) with $k = 1$ and $k = 2$, and then using (65) and (55), we obtain the following estimates:

$$h^k\overset{k}{\sigma}_{33}/G = 0(\eta^2\varepsilon), \quad k = 2, 3, \tag{68}$$

$$h^k\overset{k}{\omega} = 0(\eta^2\varepsilon), \quad k = 2, 3. \tag{69}$$

When we consider eqn (34) with $k = 2$ and $k = 3$, and make use of (67) and (68), this gives

$$h^k\overset{k}{\varepsilon}_{33} = 0(\eta^2\varepsilon), \quad k = 2, 3. \tag{70}$$

Next, the consideration of (38) with $k = 2$ and $k = 3$, together with (67), (68), (69) and (53) allows us to write

$$h^k\overset{k}{\varepsilon}_{\alpha 3} = 0(\eta^3\varepsilon), \quad k = 3, 4. \tag{71}$$

Finally, considering (32) and (33) with $k = 3$ and $k = 4$, and making use of (70) and (71) yields

$$h^k\overset{k}{\varepsilon}_{\alpha\beta} = 0(\eta^4\varepsilon), \quad k = 4, 5, \quad (\alpha, \beta = 1, 2). \tag{72}$$

It is apparent that the estimation procedure presented above allows us to write

$$e_{\alpha\beta} = \sum_{k=0}^{3} z^k/k!\overset{k}{\varepsilon}_{\alpha\beta} + 0(\eta^4\varepsilon), \quad (\alpha, \beta = 1, 2), \tag{73}$$

$$e_{\alpha 3} = \sum_{k=0}^{2} z^k/k!\overset{k}{\varepsilon}_{\alpha 3} + 0(\eta^3\varepsilon). \tag{74}$$

The strain field (73)–(74) is a starting point of the construction of the plate theory.

4. CONSTRUCTION OF PLATE THEORY

The strain field given by (73) and (74) can be further transformed without any change in the accuracy. When the conditions that the transverse shear strains $e_{\alpha 3}$ vanish on the top and bottom surfaces of the plate are applied to (74), we obtain

$$h\overset{1}{\varepsilon}_{\alpha 3} = 0(\eta^3 \varepsilon), \tag{75}$$

$$h^2/8\overset{2}{\varepsilon}_{\alpha 3} = -\overset{0}{\varepsilon}_{\alpha 3} + 0(\eta^3 \varepsilon). \tag{76}$$

It follows from these relations and (74) that

$$e_{\alpha 3} = (1 - 4z^2/h^2)\overset{0}{\varepsilon}_{\alpha 3} + 0(\eta^3 \varepsilon). \tag{77}$$

Instead of $e_{\alpha 3}$ we introduce a new measure of the shearing strain $\bar{e}_{\alpha 3}$ defined by

$$\bar{e}_{\alpha 3} = e_{\alpha 3} \int_{-h/2}^{h/2} e_{\alpha 3}/\overset{0}{\varepsilon}_{\alpha 3}\, dz \Big/ \int_{-h/2}^{h/2} e_{\alpha 3}^2/\overset{0}{\varepsilon}{}_{\alpha 3}^2\, dz. \tag{78}$$

Thus

$$\bar{e}_{\alpha 3} = 5/4(1 - 4z^2/h^2)\overset{0}{\varepsilon}_{\alpha 3} + 0(\eta^3 \varepsilon). \tag{79}$$

Considering (42) with $k = 0$ and $k = 1$, and by the use of (75), (76), (59), (63), (64) and (53), we have

$$\overset{0}{\varepsilon}_{33,\alpha}/A_\alpha = -v\overset{0}{\omega}_{\alpha\beta}/A_\beta + 0.5v\overset{0}{X}_\alpha/G + 0(\eta^3\varepsilon/h), \quad (\alpha \neq \beta), \tag{80}$$

$$\overset{1}{\varepsilon}_{33,\alpha}/A_\alpha = -4v/h^2\overset{0}{\varepsilon}_{\alpha 3} + 0(\eta^3\varepsilon/h^2). \tag{81}$$

After applying (80) and (81) to (32) and (33), where $k = 0$ and $k = 1$, and estimating these equations with the help of (54) and (57), and then using (76), we arrive at

$$h^2\overset{2}{\varepsilon}_{\alpha\beta} = 0(\eta^4\varepsilon), \quad (\alpha, \beta = 1, 2), \tag{82}$$

$$\overset{3}{\varepsilon}_{\alpha\alpha} = -8/h^2(1 - v/2)(\overset{0}{\varepsilon}_{\alpha 3,\alpha}/A_\alpha - \overset{0}{\varepsilon}_{\beta 3}/\rho_\alpha) + 0(\eta^4\varepsilon/h^3), \quad (\alpha \neq \beta), \tag{83}$$

$$\overset{3}{\varepsilon}_{12} = -8/h^2(1 - v/2)(B_1^+\overset{0}{\varepsilon}_{23} + B_2^+\overset{0}{\varepsilon}_{13}) + 0(\eta^4\varepsilon/h^3). \tag{84}$$

It follows that the strains parallel to the middle surface are of the form

$$e_{\alpha\beta} = \overset{0}{\varepsilon}_{\alpha\beta} + z\overset{1}{\varepsilon}_{\alpha\beta} + z^3/6\overset{3}{\varepsilon}_{\alpha\beta} + 0(\eta^4\varepsilon), \quad (\alpha, \beta = 1, 2), \tag{85}$$

where $\overset{3}{\varepsilon}_{\alpha\beta}$ is defined by (83) and (84).

The displacement field which is needed for the complete representation of the strains (79) and (85) is of the form

$$u_\alpha = \overset{0}{u}_\alpha + z\overset{1}{u}_\alpha + z^3/6\overset{3}{u}_\alpha, \tag{86}$$

$$u_3 = \overset{0}{u}_3. \tag{87}$$

In view of (30), (81) and (76), we have

$$\overset{3}{u}_\alpha = -8/h^2(1-\nu/2)\overset{0}{\varepsilon}_{\alpha 3}+0(\eta^3\varepsilon/h^2). \tag{88}$$

It should be noticed that only five parameters of the displacement field ($\overset{0}{u}_1, \overset{0}{u}_2, \overset{1}{u}_1, \overset{1}{u}_2, \overset{0}{u}_3$) are independent.

Introducing the strain field (79) and (85) into the three-dimensional strain energy and after performing the integration with respect to z, the two-dimensional form of this energy is

$$\begin{aligned} U_s = \tfrac{1}{2}\int_\Omega \{ & E/(1-\nu^2)[h(\overset{0}{\varepsilon}{}^2_{11}+2\nu\overset{0}{\varepsilon}_{11}\overset{0}{\varepsilon}_{22}+\overset{0}{\varepsilon}{}^2_{22})+h^3/12(\overset{1}{\varepsilon}{}^2_{11}+2\nu\overset{1}{\varepsilon}_{11}\overset{1}{\varepsilon}_{22}+\overset{1}{\varepsilon}{}^2_{22}) \\ & +h^5/240(\overset{1}{\varepsilon}_{11}\overset{3}{\varepsilon}_{11}+\nu\overset{1}{\varepsilon}_{11}\overset{3}{\varepsilon}_{22}+\nu\overset{1}{\varepsilon}_{22}\overset{3}{\varepsilon}_{11}+\overset{1}{\varepsilon}_{22}\overset{3}{\varepsilon}_{11})+h^7/16{,}128(\overset{3}{\varepsilon}{}^2_{11}+2\nu\overset{3}{\varepsilon}_{11}\overset{3}{\varepsilon}_{22}+\overset{3}{\varepsilon}{}^2_{22})] \\ & +G[h\overset{0}{\varepsilon}{}^2_{12}+h^3/12\overset{1}{\varepsilon}{}^2_{12}+h^5/240\overset{1}{\varepsilon}_{12}\overset{3}{\varepsilon}_{12}+h^7/16{,}128\overset{3}{\varepsilon}{}^2_{12}+5/6h(\overset{0}{\varepsilon}{}^2_{13}+\overset{0}{\varepsilon}{}^2_{23})]+0(Eh\eta^4\varepsilon^2)\}\,\mathrm{d}\Omega, \end{aligned} \tag{89}$$

where $\mathrm{d}\Omega$ is an elementary area of the middle surface.

The resultants in terms of the strain components are defined by

$$N_{11} = \partial U_s/\partial\overset{0}{\varepsilon}_{11}, \ldots, \tag{90}$$

The equations of motion and the appropriate boundary conditions can be derived from the principle of virtual work for the elastodynamic problem

$$\int_0^t (\delta U - \delta T)\,\mathrm{d}t = 0, \tag{91}$$

where

$$\begin{aligned} U = \int_\Omega [& N_{11}\overset{0}{\varepsilon}_{11}+N_{22}\overset{0}{\varepsilon}_{22}+N_{12}\overset{0}{\varepsilon}_{12}+M_{11}\overset{1}{\varepsilon}_{11}+M_{22}\overset{1}{\varepsilon}_{22}+M_{12}\overset{1}{\varepsilon}_{12}+\overset{3}{M}_{11}\overset{3}{\varepsilon}_{11}+\overset{3}{M}_{22}\overset{3}{\varepsilon}_{22}+\overset{3}{M}_{12}\overset{3}{\varepsilon}_{12} \\ & +N_{13}\overset{0}{\varepsilon}_{13}+N_{23}\overset{0}{\varepsilon}_{23}+\lambda_1(\overset{3}{u}_1+c\overset{0}{\varepsilon}_{13})+\lambda_2(\overset{3}{u}_2+c\overset{0}{\varepsilon}_{23})]\,\mathrm{d}\Omega, \end{aligned} \tag{92}$$

$$c = 4(2-\nu)/h^2, \tag{93}$$

$$\begin{aligned} T = \tfrac{1}{2}\int_\Omega \rho[& h(\overset{0}{u}{}^2_{1,t}+\overset{0}{u}{}^2_{2,t}+\overset{0}{u}{}^2_{3,t})+h^3/12(\overset{1}{u}{}^2_{1,t}+\overset{1}{u}{}^2_{2,t}) \\ & +h^5/240(\overset{1}{u}_{1,t}\overset{3}{u}_{1,t}+\overset{1}{u}_{2,t}\overset{3}{u}_{2,t})+h^7/16{,}128(\overset{3}{u}{}^2_{1,t}+\overset{3}{u}{}^2_{2,t})]\,\mathrm{d}\Omega. \end{aligned} \tag{94}$$

Here λ_1 and λ_2 are the Lagrange multipliers and ρ is the mass density of the plate.

From (91) we obtain the following equations of motion:

$$G_\alpha(N) = \rho h\overset{1}{u}_{\alpha,tt}, \tag{95}$$

$$[A_2(N_{13}+c\lambda_1)]_{,1}+[A_1(N_{23}+c\lambda_2)]_{,2} = \rho h\overset{0}{u}_{3,tt}, \tag{96}$$

$$G_\alpha(M)-(N_{\alpha 3}+c\lambda_\alpha) = \rho h^3/12(\overset{1}{u}_{\alpha,tt}+h^2/40\overset{3}{u}_{\alpha,tt}), \tag{97}$$

$$G_\alpha(\overset{3}{M})-\lambda_\alpha = \rho h^5/480(\overset{1}{u}_{\alpha,tt}+10h^2/336\overset{3}{u}_{\alpha,tt}), \tag{98}$$

with the subsidiary conditions

$$\overset{3}{u}_\alpha + c\overset{0}{\varepsilon}_{\alpha 3} = 0, \tag{99}$$

where

$$G_\alpha(F) = F_{\alpha\alpha,\alpha}/A_\alpha - (F_{\alpha\alpha} - F_{\beta\beta})/\rho_\beta + F_{\alpha\beta,\beta}/A_\beta - 2F_{\alpha\beta}/\rho_\alpha, \quad (\alpha \neq \beta). \tag{100}$$

It follows that the boundary conditions are equivalent to the specification of the displacement boundary conditions or the resultant boundary conditions:

$$\begin{array}{ll}
\alpha_1 = \text{constant} & \alpha_2 = \text{constant} \\
\overset{0}{u}_1 \text{ or } N_1 & \overset{0}{u}_1 \text{ or } N_{12} \\
\overset{0}{u}_2 \text{ or } N_{12} & \overset{0}{u}_2 \text{ or } N_2 \\
\overset{0}{u}_3 \text{ or } N_3 + c(\lambda_1 + \overset{3}{M}_{12,2}/A_2) & \overset{0}{u}_3 \text{ or } N_3 + c(\lambda_2 + \overset{3}{M}_{12,1}/A_1) \\
\overset{1}{u}_1 \text{ or } M_1 & \overset{1}{u}_1 \text{ or } M_{12} - c\overset{3}{M}_{12} \\
\overset{1}{u}_2 \text{ or } M_{12} - c\overset{3}{M}_{12} & \overset{1}{u}_2 \text{ or } M_2 \\
\overset{3}{u}_1 \text{ or } \overset{3}{M}_1 & \overset{3}{u}_2 \text{ or } \overset{3}{M}_2.
\end{array} \tag{101}$$

If the edge curve does not coincide with the lines α^1 = constant or α^2 = constant these conditions have to be appropriately transformed.

The Lagrange multipliers λ_1 and λ_2 can be eliminated from the equations of motion in a similar way to that in which the shear forces are eliminated from the equations of the classical theory of plates. Moreover, using the subsidiary condition (101) the parameters $\overset{3}{u}_\alpha$ can also be eliminated from these equations and the boundary conditions.

5. SUMMARY AND CONCLUSIONS

Using the estimates of the strain field the new refined version of the linear theory of isotropic plates is constructed. The theory is expressed in terms of five independent displacement parameters and ensures the relative accuracy of the strain energy of the order of η^4 in comparison to the accuracy of the order of η^2 in the classical theory of plates. The accuracy is ensured provided the variability of the rotation $\overset{0}{\omega}$ about the normal vector n on the middle surface is appropriately lower than that of the two-dimensional components of strains. It follows that if the plate is in the bending state the accuracy is always of the order of η^4. The equations of motion are derived with the help of the variationally consistent method.

From the present study we can conclude that the consistent higher-order theory is based on the displacement field (86)–(87) together with the relation (88). In some theories (Mushtari, 1959; Jemielita, 1975; Lewiński, 1987), additional terms in the displacement field were included. It follows that in the sense of the energy criterion, the inclusion of those additional terms does not improve the present theory. Furthermore, the elimination of the parameters $\overset{3}{u}_\alpha$ has to be done according to eqn (88). In view of the present analysis, the methods of elimination of $\overset{3}{u}_\alpha$ proposed earlier by Mushtari (1959), Kączkowski (1968), Jemielita (1975), Reissner (1975), Levinson (1980) and Lewiński (1987) are incorrect. The coefficient $(1-\nu/2)$ in (88) cannot be approximated by 1.

The form of eqn (88) can be confirmed by the use of the results of Lur'e (1955). In that analysis a plate which is free from loads on the top and bottom surfaces and subjected to antisymmetric loads (so-called bending loads) on the edge of the plate is considered. The exact three-dimensional solution of the biharmonic stress state for u_α reads (Lur'e, 1955, section 4)

$$u_\alpha = -z\overset{0}{u}_{3,x} + [(2-\nu)z^3/6 - h^2 z/4]/(1-\nu)(\nabla^2 \overset{0}{u}_3)_{,x},$$

where x—an in-plane coordinate, ∇^2—the Laplacian, $\overset{0}{u}_3$—the normal displacement of the middle surface of the plate which satisfies the biharmonic equation. This type of solution was also shown by Cheng (1979).

When we put

$$\overset{1}{u}_\alpha = -\overset{0}{u}_{3,x} - h^2/[4(1-\nu)](\nabla^2 \overset{0}{u}_3)_{,x},$$

it gives

$$u_\alpha = z\overset{1}{u}_\alpha + z^3/6\overset{3}{u}_\alpha,$$

where

$$\overset{3}{u}_\alpha = -8/h^2(1-\nu/2)\overset{0}{\varepsilon}_{\alpha 3}.$$

We notice that the above equation has the same form as (88). The additional two stress states which complete the solution of this problem are of appropriately lower order in comparison to the biharmonic stress state (Arksentian and Vorovich, 1963). Moreover, these two types of stress states vanish in the interior domain of the plate.

It follows that the governing equations of the present theory differ from those proposed earlier, though the order of the equations of motion is the same as that of Reddy (1984). The problem of anisotropic plates with increased transverse shear deformability will be discussed in a forthcoming paper.

In the analysis of the estimations (Section 3), the effect of loads distributed over the faces of the plate can be easily incorporated into the appropriate equations as it was shown for the case of shells in Blocki (1982).

The present theory has been used in an analysis of free vibrations of discs. This problem is the subject of Part II of this paper (Rządkowski, 1992).

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